XLIII. *A letter from the late Reverend Mr.* Thomas Bayes, *F. R. S. to* John Canton, *M. A. and F. R. S.*

SIR,

Read Nov. 24, 1763. If the following observations do not seem to you to be too minute, I should esteem it as a favour, if you would please to communicate them to the Royal Society.

It has been asserted by some eminent mathematicians, that the sum of the logarithms of the numbers $1 . 2 . 3 . 4$ &c to z, is equal to $\frac{1}{2}\log . c + \overline{z + \frac{1}{2}} \times \log . z$ lessened by the series $z - \frac{1}{12z} + \frac{1}{360z^3} - \frac{1}{1260z^5} + \frac{1}{1680z^7} - \frac{1}{1182z^9} +$ &c. if c denote the circumference of a circle whose radius is unity. And it is truue that this expression will very nearly approach to the value of that sum when z is large, and you take in only a proper number of the first terms of the foregoing series; but the whole series can never properly express any quantity at all; because after the 5th term the coefficients begin to increase, and they afterwards increase at a greater rate than what can be comprehended by the increase of the powers of z, though z represent a number ever so large; as will be evident by considering the following manner in which the coefficients of that series may be formed. Take $a = \frac{1}{12}$, $5b = a^2$, $7c = 2ba$, $9b = 2ca + b^2$, $11e = 2da + 2cb$, $13f = 2ea + 2db + c^2$, $15g = 2fa + 2eb + 2dc$, and so on; then take A $= a$, B $= 2b$, C $= 2 \times 3 \times 4c$, D $= 2 \times 3 \times 4 \times 5 \times 6d$, E $= 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8e$ and so on, and A, B, C, D, E, F, &c. will be the coefficients of the foregoing series : from whence it easily follows, that if any term in the series after the 3 first be called y, and its distance from the first term n, the term immediately following will be greater than $\frac{n \times \overline{2n-1}}{6n+9} \times \frac{y}{z^2}$. Whereforeatlength the subsequent terms of this series are greater than the preceding ones, and increase in infinitum, and therefore the whole series can have no ultimate value whatsoever.

Much less can that series have any ultimate value, which is deduced from it by taking $z = 1$, and is supposed to be equal to the logarithm of the square root of the periphery of a circle whose whole radius is unity; and what is said concerning the foregoing series is true, and appears to be so, much in the same manner, concernign the series for finding out the sum of the logarithms of the odd numbers $3 . 5 . 7$ &c. . . . z and those that are given for finding out the sum of the infinite proggressions, in which the several terms have the same numerator whilst their denominators are any certain power of numbers increasing in arithmetical proportion. But it is needless particularly to insist upon these, because one instance is sufficient to shew that those methods are not to be depended upon, form which a conclusion follows that is not exact.

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SOME REMARKS CONCERNING BAYES' NOTE ON THE USE OF CERTAIN DIVERGENT SERIES

The same volume of the Philosophical Trasactions (1763) that contains Bayes' essay "towards solving a problem in the doctrine of chances," contains also a note on divergent series, barely covering two pages, yet describing the essentials, save for a discussion of the remainder term, of what we now call asymptotic expansions. At the foot of page 401 of the essay Price refers to a note by Bayes on series, and it is this note that we now have in mind. The occasion was the divergent series for the factorials in the term Ea^pb^q, or in more familiar notation, the term $\binom{n}{q}a^{n-q}b^q$ in the binomial expansion of $(a+b)^n$. For his illustration Bayes dealt with the series

$$\log z! \sim \log\sqrt{2\pi} + (z+\tfrac{1}{2})\log z \\ - \left\{ z - \frac{1}{1.2z}B_1 + \frac{1}{3.4z^3}B_3 - \frac{1}{5.6z^5}B_2 + - \dots \right\}$$

commonly known as Stirling's formula, and used by DeMoivre in his derivation of the binomial (1733). It seems more probable that Bayes had DeMoivre and Stirling in mind when he spoke of "some eminent mathematicians," as several clues indicate: *first,* the symbol c for 2π was commonly used by DeMoivre and Stirling; *second,* of all the series that Bayes could have chosen for his illustration, the one he actually used in the one to which these two men had devoted so much study; *third,* Price, who knew Bayes' mind as well as anyone, speaks of Stirling on page 401 of the essay with the same adjective; *fourth,* in the *Approximatio* DeMoivre in the words *attamen tarditas convergientiae me deterruerat quominus longius procedurem* relates his struggle with the series

$$B = -\frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} + \&\text{c.}$$

which is none other than the result of putting $z = 1$ in the complementary terms of the expansion for $\log z!$.

Tarditas convergientae: apparently neither he nor Stirling realized that his series is only symbolic for $\log\sqrt{2\pi}$, and that it does not converge at all. If they had written down one more term (1/1188) they would have perceived *divergence.* But we must remember that convergence was not carefully defined as a limiting process in those days, though Stirling in his *Methodus Differentialis* gave the earliest test for convergence.

At the time Bayes dies, Euler's *Institutiones Calculi Differenitalis* had been in existence six years. Therein on page 465 Euler remarks on the divergence of the factorial series for $z = 1$, whereupon for the evaluation of the constant ($\log\sqrt{2\pi}$) he resorts to Wallis' product, as Stirling had done before him. It is not possible to judge from this or his earlier writings whether Euler was aware that this series diverges for all values of z however large; neither is it clear whether he recognised the eventual divergence of the "Euler-Maclaurin summation formula" for functions not polynomials. It may therefore be that this note by Bayes contains the first recognition of the existence of an optimum term, i.e., of the asymptotic character of the series, and one can not help but wonder when he actually wrote it; in particular, whether he perceived it from anything that Euler had said. The manuscript that Price sent in to the Royal Society is not dated, and whether Price sent Bayes' original paper to the Royal Society, or copied it omitting the date, no one seems to know. Many enquiries received by the Secretaries of the Royal Society give evidence of curiosity before ours; but regardless of the answer, the note commands respect for the mathematical attainments of the author of the essay.

W. Edwards Deming